# Popular Computing

The world's only magazine devoted to the art of computing

## VALUES OF CHI-SQUARE

| | Probability of a Larger Value of Chi-Square | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| D.f. | 0.99 | 0.98 | 0.95 | 0.90 | 0.80 | 0.70 | 0.50 | 0.30 | 0.20 | 0.10 | 0.05 | 0.02 | 0.01 |
| 1 | .000 | .001 | .004 | .016 | .064 | .148 | .455 | 1.074 | 1.642 | 2.706 | 3.841 | 5.412 | 6.635 |
| 2 | .020 | .040 | .103 | .211 | .446 | .713 | 1.386 | 2.408 | 3.219 | 4.605 | 5.991 | 7.824 | 9.210 |
| 3 | .115 | .185 | .352 | .584 | 1.005 | 1.424 | 2.366 | 3.665 | 4.642 | 6.251 | 7.815 | 9.837 | 11.341 |
| 4 | .297 | .429 | .711 | 1.064 | 1.649 | 2.195 | 3.357 | 4.878 | 5.989 | 7.779 | 9.488 | 11.668 | 13.277 |
| 5 | .554 | .752 | 1.145 | 1.610 | 2.343 | 3.000 | 4.351 | 6.064 | 7.289 | 9.326 | 11.070 | 13.388 | 15.086 |
| 6 | .872 | 1.134 | 1.635 | 2.204 | 3.070 | 3.828 | 5.348 | 7.231 | 8.558 | 10.645 | 12.592 | 15.033 | 16.812 |
| 7 | 1.239 | 1.564 | 2.167 | 2.833 | 3.822 | 4.671 | 6.346 | 8.383 | 9.803 | 12.017 | 14.067 | 16.622 | 18.475 |
| 8 | 1.646 | 2.032 | 2.733 | 3.490 | 4.594 | 5.527 | 7.344 | 9.524 | 11.030 | 13.362 | 15.507 | 18.168 | 20.090 |
| 9 | 2.088 | 2.532 | 3.325 | 4.168 | 5.380 | 6.393 | 8.343 | 10.656 | 12.242 | 14.684 | 16.919 | 19.679 | 21.666 |
| 10 | 2.558 | 3.059 | 3.940 | 4.865 | 6.179 | 7.267 | 9.342 | 11.781 | 13.442 | 15.987 | 18.307 | 21.161 | 23.209 |
| 11 | 3.053 | 3.609 | 4.575 | 5.578 | 6.989 | 8.148 | 10.341 | 12.899 | 14.631 | 17.275 | 19.675 | 22.618 | 24.725 |
| 12 | 3.571 | 4.178 | 5.226 | 6.304 | 7.807 | 9.034 | 11.340 | 14.011 | 15.812 | 18.549 | 21.026 | 24.054 | 26.217 |
| 13 | 4.107 | 4.765 | 5.892 | 7.042 | 8.634 | 9.926 | 12.340 | 15.119 | 16.985 | 19.812 | 22.362 | 25.472 | 27.688 |
| 14 | 4.660 | 5.368 | 6.571 | 7.790 | 9.467 | 10.821 | 13.339 | 16.222 | 18.151 | 21.064 | 23.685 | 26.873 | 29.141 |
| 15 | 5.229 | 5.985 | 7.261 | 8.547 | 10.307 | 11.721 | 14.339 | 17.322 | 19.311 | 22.307 | 24.996 | 28.259 | 30.578 |
| 16 | 5.812 | 6.614 | 7.962 | 9.312 | 11.152 | 12.624 | 15.338 | 18.418 | 20.465 | 23.542 | 26.296 | 29.633 | 32.000 |
| 17 | 6.408 | 7.255 | 8.672 | 10.085 | 12.002 | 13.531 | 16.338 | 19.511 | 21.615 | 24.769 | 27.587 | 30.995 | 33.409 |
| 18 | 7.015 | 7.906 | 9.390 | 10.865 | 12.857 | 14.440 | 17.338 | 20.601 | 22.760 | 25.989 | 28.869 | 32.346 | 34.805 |
| 19 | 7.633 | 8.567 | 10.117 | 11.651 | 13.716 | 15.352 | 18.338 | 21.689 | 23.900 | 27.204 | 30.144 | 33.687 | 36.191 |
| 20 | 8.260 | 9.237 | 10.851 | 12.443 | 14.578 | 16.266 | 19.337 | 22.775 | 25.038 | 28.412 | 31.410 | 35.020 | 37.566 |
| 21 | 8.897 | 9.915 | 11.591 | 13.240 | 15.445 | 17.182 | 20.337 | 23.858 | 26.171 | 29.615 | 32.671 | 36.343 | 38.932 |
| 22 | 9.542 | 10.600 | 12.338 | 14.041 | 16.314 | 18.101 | 21.337 | 24.939 | 27.301 | 30.813 | 33.924 | 37.659 | 40.289 |
| 23 | 10.196 | 11.293 | 13.091 | 14.848 | 17.187 | 19.021 | 22.337 | 26.018 | 28.429 | 32.007 | 35.172 | 38.968 | 41.638 |
| 24 | 10.856 | 11.992 | 13.848 | 15.659 | 18.062 | 19.943 | 23.337 | 27.096 | 29.553 | 33.196 | 36.415 | 40.270 | 42.980 |
| 25 | 11.524 | 12.697 | 14.611 | 16.473 | 18.940 | 20.867 | 24.337 | 28.172 | 30.675 | 34.382 | 37.652 | 41.566 | 44.314 |
| 26 | 12.198 | 13.409 | 15.379 | 17.292 | 19.820 | 21.792 | 25.336 | 29.246 | 31.795 | 35.563 | 38.885 | 42.856 | 45.642 |
| 27 | 12.879 | 14.125 | 16.151 | 18.114 | 20.703 | 22.719 | 26.336 | 30.319 | 32.912 | 36.741 | 40.113 | 44.140 | 46.963 |
| 28 | 13.565 | 14.847 | 16.928 | 18.939 | 21.588 | 23.647 | 27.336 | 31.391 | 34.027 | 37.916 | 41.337 | 45.419 | 48.278 |
| 29 | 14.256 | 15.574 | 17.708 | 19.768 | 22.475 | 24.577 | 28.336 | 32.461 | 35.139 | 39.087 | 42.557 | 46.693 | 49.588 |
| 30 | 14.953 | 16.306 | 18.493 | 20.599 | 23.364 | 25.508 | 29.336 | 33.530 | 36.250 | 40.256 | 43.773 | 47.962 | 50.802 |

# CHI-SQUARED...and a problem ▶

# CHI SQUARED

The statistical test called chi-squared provides a
means of testing the relationship between two sets of
numbers, to determine the "goodness of fit" of one set
against another.   Usually, for this use of chi-squared,
one set of numbers is theoretical, and the other set is
observed.

The standard table of chi-squared values is given
here.   It is probably the most widely reproduced table
in the world.   Wilks, in his biography of Karl Pearson
in the Brittanica, says "One of Pearson's greatest
contributions to statistical theory was the chi-square
test of goodness of fit which was introduced in 1900."
The table was calculated by R. A. Fisher and has been in
use unchanged for well over half a century.

Consider, for example, the tabulations shown in
Table A of 600 tosses of a die.   A cubical die is a
random number generator that is supposed to furnish the
numbers from 1 to 6 with equal likelihood.   Thus, in
600 tosses, the most probable distribution is 100 of each
number.   The probability of obtaining that exact distrib-
ution, however, is extremely small:

$$\frac{600!}{(100!)^6} \; (1/6)^{600} \; = \; .000000246328583$$

In other words, for a random event, we expect a certain
amount of random variation to take place.

Now, not much knowledge of statistics is needed to
see that the distribution in column A is suspect; the
numbers are too close to the theoretical frequencies.
We expect observed results that are similar to the
theoretical frequencies, but which are not too close to
them.

| Number appearing on the die | Theoretical frequency | Trial A observed frequencies | Trial B observed frequencies | Trial C observed frequencies | Trial D observed frequencies |
|---|---|---|---|---|---|
| 1 | 100 | 99 | 118 | 96 | 108 |
| 2 | 100 | 102 | 81 | 104 | 92 |
| 3 | 100 | 103 | 112 | 105 | 94 |
| 4 | 100 | 97 | 86 | 95 | 111 |
| 5 | 100 | 98 | 83 | 96 | 90 |
| 6 | 100 | 101 | 120 | 104 | 105 |
| totals | 600 | 600 | 600 | 600 | 600 |

Table A: Various distributions of 600 tosses of a single die.

Similarly, the observed frequencies in column B are too far away from the theoretical. For both columns A and B, the chi-squared test tells us, as it should, that we do not have a good fit.

The formula for goodness-of-fit is:

$$(CS) \qquad \sum \frac{(f_o - f_t)^2}{f_t} = \chi^2$$

where $f_o$ is an observed frequency, $f_t$ is a theoretical frequency, and the summation is to be taken over all values. Thus, in column D of Table A, the calculation of chi-squared is as follows:

| $f_t$ | $f_o$ | $f_o - f_t$ | $(f_o - f_t)^2$ |
|-------|-------|-------------|------------------|
| 100   | 108   | +8          | 64               |
| 100   | 92    | -8          | 64               |
| 100   | 94    | -6          | 36               |
| 100   | 111   | +11         | 121              |
| 100   | 90    | -10         | 100              |
| 100   | 105   | +5          | 25               |

and chi-squared is the sum of the numbers in the last column, each divided by the theoretical value of 100. The result is 4.10.

The chi-squared table is arranged in rows for each possible number of degrees of freedom, up to 30 (and beyond 30 degrees of freedom, one should not be using chi-squared).    Here, we assume that the sum for each distribution is fixed at 600, which means that only five of the six numbers have any freedom (that is, once five of them are selected, the sixth number is determined by the sum), so the line to use in the table is D.f. = 5. We look on line 5 for the value 4.100 (seldom, in practice, does one find exactly the value one is looking for) and locate it somewhere between the 6th and 7th columns of the table (that is, between table entries 3.000 and 4.351). The headings at the top of the columns give probability levels, and we are somewhere between p = .70 and p = .50. At a rough guess, call it 53%.    (We use the table to obtain probabilities--notice that the body of the table is carried to 5 digits of precision, but we read our results from the stub of the table containing the probability levels, which is expressed with 2 digits.)

Our result, 53%, is the probability that the chi-squared value represents chance variation.    By common agreement, any value between p = .95 and p = .05 is considered not significant; that is, it represents normal variation.    Values of chi-squared greater than the ones at p = .05, or less than the ones at p = .95 are taken as significant (at the 5% level), and values outside those at p = .01 or p = .99 are taken as highly significant (that is, at the 1% level).

Look at the two extremes. If the observed and theoretical frequencies were the same, the value of chi-squared would be zero, and the table then tells us that it is virtually a certainty that chi-squared should be larger than that with random variation taking place. At the other extreme, if the observed and theoretical frequencies are far apart, then the value of chi-squared will be large, and the table may tell us that there is only one chance in a thousand that chi-squared could get that large by chance variation alone. Thus, for goodness of fit, we want values of chi-squared in the central part of the table.

Returning to our dice experiment, we can calculate the following results:

|  | A | B | C | D |
|---|---|---|---|---|
| chi-squared: | .280 | 17.140 | 1.140 | 4.100 |
|  | P > .99 | p < .01 | p = .95 | p = .60 |

Case A is, as we have already observed, highly significant-- the values are too close to the theoreticals. Case B is also highly significant, but for the opposite reason--the values are too far away from the theoreticals. Case C is just outside the p = .95 limit; it is significant at the 5% level. Case D is comfortably in the middle, and is not significant, which is to say that the data of Case D constitute a good fit.

Chi-squared is also used to test for association and independence in situations where there can be no theoretical frequencies. Consider the so-called contingency table shown as Table T. We wish to test the internal consistency of the table.

|  |  |  |  |  | row totals |
|---|---|---|---|---|---|
| 40 | 50 | 60 | 70 | 80 | 300 |
| 40 | 40 | 50 | 60 | 70 | 260 |
| 50 | 60 | 51 | 40 | 50 | 251 |
| 30 | 40 | 50 | 60 | 80 | 260 |
| 35 | 45 | 55 | 65 | 75 | 275 |

(T)

| Column totals | 195 | 235 | 266 | 295 | 355 | 1346 = G grand total |
|---|---|---|---|---|---|---|

The following table shows the distribution of the digits in the first 100,000 decimal places of pi and e. For each of these distributions, the number of degrees of freedom is 9.   The test of goodness of fit (against the theory that each entry should be 10,000) shows:

|                    | for pi                   | for e                     |
|--------------------|--------------------------|---------------------------|
|                    | chi-squared = $4.093$    | chi-squared = $17.682$    |
|                    | $p \cong .91$            | $p \cong .04$             |

| digit | number occurring in pi | number occurring in e |
|-------|------------------------|-----------------------|
| 0     | 9999                   | 9885                  |
| 1     | 10137                  | 10264                 |
| 2     | 9908                   | 9855                  |
| 3     | 10025                  | 10035                 |
| 4     | 9971                   | 10039                 |
| 5     | 10026                  | 10034                 |
| 6     | 10029                  | 10183                 |
| 7     | 10025                  | 9875                  |
| 8     | 9978                   | 9967                  |
| 9     | 9902                   | 9863                  |

First, let us examine two other contingency tables. As before, no knowledge of statistics is needed to see that Table V is highly consistent; that is, all the rows exhibit the same distribution, with minor variation. Similarly, one can readily see that Table W is quite inconsistent; that is, the row distributions seem to be drawn from entirely different sources.

| | | | | |
|---|---|---|---|---|
| 40 | 50 | 60 | 70 | 80 |
| 41 | 51 | 63 | 72 | 85 |
| 39 | 51 | 59 | 69 | 78 |
| 41 | 48 | 61 | 71 | 81 |
| 40 | 60 | 50 | 70 | 80 |

**(V)**

| | | | | |
|---|---|---|---|---|
| 40 | 50 | 60 | 70 | 80 |
| 80 | 40 | 50 | 60 | 70 |
| 60 | 50 | 40 | 80 | 70 |
| 70 | 60 | 50 | 40 | 80 |
| 90 | 10 | 20 | 30 | 40 |

**(W)**

As with the test for goodness of fit, the question
is where to draw the line between these two extremes.   We
can tolerate some variation (indeed, if we are observing
some natural phenomenon, or counting responses, we expect
to find some variation), but not too much.   In this case,
we do not have any theoretical frequencies to match against.
We want to apply chi-squared to Table T, using formula CS,
but first we must fabricate theoretical frequencies for
each of the 25 cells of Table T.

We reason that the row and column totals reflect the
overall conditions of the table, and we calculate a
theoretical frequency for the cell in row i, column j
as follows:

$$\textbf{(Y)} \qquad f_{t_{ij}} = \frac{R_i C_j}{G}$$

That is, for each cell, we multiply its row total by its
column total and divide by the grand total.   For example,
in Table T, the theoretical frequency for the central cell
of the table is:

$$\frac{251 \cdot 266}{1346} = 49.60$$

We can make the following observations:

1.  If all the rows (or all the columns) were alike, the calculated theoretical frequencies would be the same as the observed frequencies.    In Table V, the theoretical frequencies will come out to be almost the same as the observed frequencies.

2.  If the table is clearly unassociated, as in Table W, the theoretical frequencies will be quite different from most of the observed freqencies.    For example, the theoretical frequencies for the middle row of table W are:

$$73.4 \qquad 45.3 \qquad 47.5 \qquad 60.4 \qquad 73.4$$

3.  The sum of all the theoretical frequencies will be the same as the sum of all the observed frequencies; namely, the grand total, G.

4.  The number of degrees of freedom for a m x n contingency table is $(m - 1)(n - 1)$.    For Table T, this is 16.

We thus have the necessary mechanism to apply chi-squared to Table T or similar tables.    As derived so far, this means calculating 25 theoretical frequencies and then applying formula CS, which calls for another 25 calculations.    This is all right (computers are made to perform tedious calculations), but quite inefficient. Notice that we will have to go over the data of the table three times:  once to obtain the row and column totals; once to calculate the theoretical frequencies; and once to evaluate formula CS.

Formula CS can be expanded and simplified, and combined with the formula for each theoretical frequency, to reach this formula for chi-squared for a contingency table:

$$G \left[ \sum \frac{f_o^2}{R \cdot C} - 1 \right]$$

For each element in the table, we square it and divide by the product of its row and column total; sum these values for all the cells of the table; subtract one; and multiply by G.    By the use of this transformation, we need to go over the data only twice, and perform much less arithmetic for the same result.    For Table T, the result is 26.417 for chi-squared, and the table then gives a value for p of slightly under .05.

Which brings us to our Problem.

⇩  ⇩  ⇩  ⇩
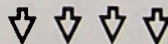
Table T was carefully constructed to produce a value of chi-squared of 26.417, which is close to the critical value (26.296) for 16 degrees of freedom.    Which entries in Table T should be altered by how much to yield a value of chi-squared of <u>exactly</u> 26.296?

# Problem Solution

Problem 218 (Fun With Equations) in issue No. 57 was the following:

Find the positive real root of

$$Ax^3 + x^2 + x - 1 = 0$$

starting with A = 1. The root thus found is to be multiplied by a factor F, and the product replaces A, and the root of this new equation is found. This process repeats (with F fixed) until it converges. The problem involved investigating the behavior of the roots for various values of F.

Robert Hall, Arleta, California, did an extensive analysis of this problem. When F = 1, the process converges to .569840291, which is to say that the equation

$$.569840291x^3 + x^2 + x - 1 = 0$$

has .569840291 for a root.

As F approaches zero, the equation tends toward

$$x^2 + x - 1 = 0$$

and the root of the converging process approaches the value .61803399. At the other extreme, when F becomes very large, the process tends toward

$$x = \frac{1}{\sqrt[4]{F}}$$

which is to say that the root becomes asymptotic to zero.

So Problem 218 can be considered solved, with a set of results showing the roots as a function of F, as in the graph.

We are thus led to a nice new problem. The curve of the results is well defined. It exists for all values of F greater than or equal to zero, and it is continuous. What is _its_ equation?
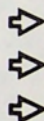
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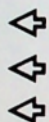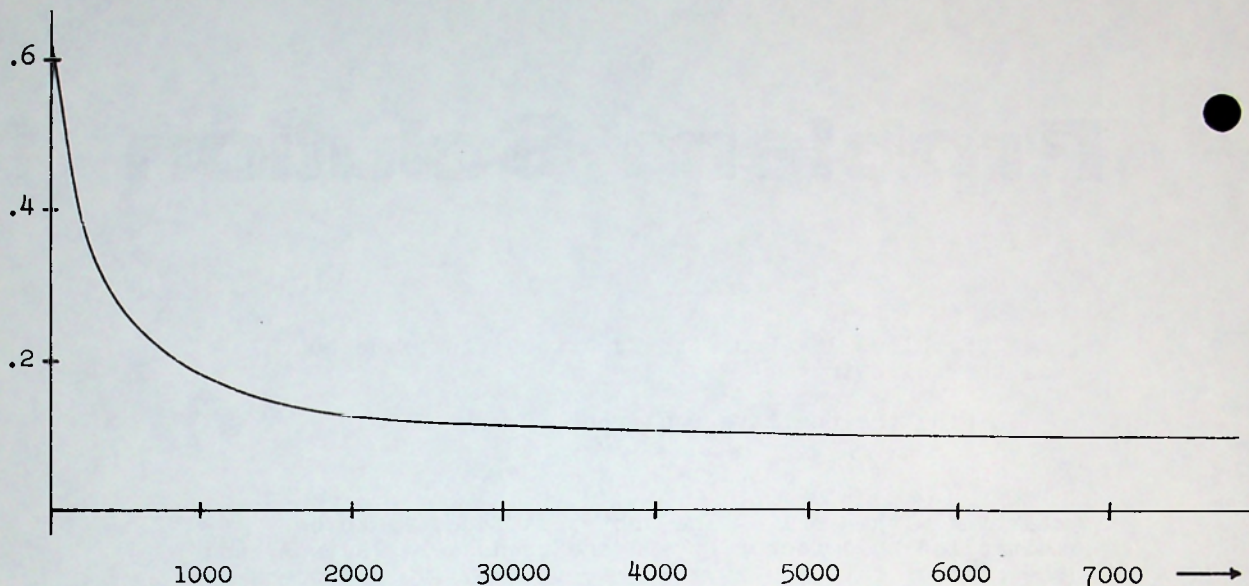
Some roots of the cubic (after convergence):

| F | x |
|---|---|
| .1 | .6117528184 |
| .5 | .5905066120 |
| 1.0 | .5698402910 |
| 2 | .5395022822 |
| 3 | .5173737435 |
| 4 | .5 |
| 5 | .4857330395 |
| 6 | .4736552027 |
| 9 | .4457985023 |
| 10 | .4384033879 |
| 20 | .3892644673 |
| 30 | .3608872303 |
| 40 | .3412320144 |
| 50 | .3263477776 |
| 100 | .2825731904 |
| 500 | .1977248888 |
| 1000 | .1683522723 |
| 5000 | .1149131327 |
| 10000 | .0972187858 |
| 20000 | .0821543006 |
| 50000 | .0656713755 |
| 100000 | .0553930174 |
| 200000 | .0466978049 |
| 1000000 | .0313634476 |
| 2000000 | .0264090461 |

# Where Is The Numerical Methods Text?

Courses in Numerical Methods are given at many universities and, with the coming wave of personal computers, might be expected to proliferate.  These courses are <u>not</u> Numerical Analysis courses.  The differences between the two were outlined in the essay "Numerical Analysis vs. Numerical Methods" in our issue number 23 (February 1975).

Numerical Analysis (a topic in mathematics) as an academic subject goes back about 100 years.  The formal study of Numerical Methods (a topic in computing) is relatively recent; the first courses were given around 1960.  The two subjects deal with the same problems (root finding, integration, function evaluation, numerical solution of differential equations, and so on) but from widely different points of view.  In the essay cited above, the salient differences were discussed, and can be summarized as follows:

1.  In a computer, most numbers do not exist. In a binary machine, for example, the number 1/3 can not exist.  The number pi cannot exist in any machine.

2.  The number system usually used (scientific notation) is not uniformly dense.

3.  There are no infinite processes in computing, and hence, for example, the use of (truncated) infinite series can introduce significant differences from the results suggested by the series.

4.  The concept of continuity is fundamental to many mathematical processes; it seldom exists in computing, because of the discrete digital nature of the computer's makeup.

Since Numerical Methods courses are computing courses, they should require that students <u>do</u> a fair amount of computing, such as:  evaluate a function; compare a Simpson integration to the trapezoidal rule; find a least squares curve fit; integrate a differential equation.  The laboratory problems should, among other things, be designed to highlight the differences listed above.  For example:

Using 8-digit floating arithmetic (that is, normal Fortran), in calculating the value of

$$\int_0^1 \sqrt{1 - x^2} \; dx$$

what is the minimum number of intervals required to achieve the value of the integral correct to four significant digits with (a) the trapezoidal rule and (b) Simpson's rule?   Perhaps a better way to look at it would be a comparison of the costs to achieve a given level of accuracy.   One should consider also the effort required to modify the program when one wishes to "double the number of intervals." Note that the notion of what constitutes "correct to four significant digits" is a topic in Numerical Methods.

In today's Numerical Methods courses, the students should be expected to have a good pocket calculator and have access to a computer.   (In at least one instance, each student in a class was furnished a pocket programmable calculator; this idea will surely spread as such machines become cheaper.)

A Numerical Methods course should have a good text book.   It is the thesis of this article that there is not as yet a suitable book.   To illustrate this idea, seven books were compared through all the topics that should be in such a text.   The books (listed alphabetically by author) are:

A    Acton, Forman

Numerical Methods That Work

Harper & Row, 1970

F    Forsythe, G.E., M.A. Malcolm, and C.B. Moler

Computer Methods for Mathematical Computation

Prentice-Hall, 1977

H    Hamming, Richard

Numerical Methods for Scientists and Engineers

McGraw-Hill, 1973 (Second edition)

M McCracken, D.D., and William Dorn

  <u>Numerical Methods and Fortran Programming</u>

  John Wiley & Sons, 1964

    The McCracken & Dorn book referred to here is now
    out of print, replaced by the Dorn & McCracken text
    <u>Numerical Methods With Fortran IV Case Studies</u>.
    The new version is greatly improved.


P Pizer, Stephen

  <u>Numerical Computing and Mathematical Analysis</u>

  Science Research Associates, 1975

S Scarborough, J. B.

  <u>Numerical Mathematical Analysis</u>

  Johns Hopkins Press, 1930

W Whittaker, Sir Edmund, and G. Robinson

  <u>The Calculus of Observations</u>

  Blackie & Son, 1946


  The books of Scarborough and Whittaker and Robinson
are included partly through nostalgia (both of them pre-
dated computers) and partly because they are representative
of the era in which much thought was given to methods of
avoiding tedious work. In addition, their exposition of
some basic topics is still worth reading.


  For the modern books, the number one topic should
be the existence of the computer and the differences the
computer milieu introduces to old problems. If the
author has not personally done extensive numerical work
with computers, his words of wisdom will have a hollow
ring. He should be aware of the pitfalls and booby
traps of numerical work that take place without being
monitored by eye (that is, in an executed program) and
be prepared to offer helpful hints on avoiding those
traps.

It does not seem unreasonable to expect a text in Numerical Methods to cover, at a minimum, the following topics:

    1.  Function evaluation.
    2.  Roots of algebraic equations.
    3.  Roots of non-algebraic equations.
    4.  Roots of simultaneous systems.
    5.  Interpolation.
    6.  Numerical integration (quadrature).
    7.  Solution of ordinary differential equations.
    8.  Curve fitting.
    9.  Error analysis.
    10. Testing of computer programs.

(Many other topics can be, and should be, included, but their choice depends on the inclination of the author and his experience in various areas.　Several methods of attack on each of the listed topics ought to be presented, with comparisons of the efficiency, stability, error analysis, and general usefulness of each method carefully explained.)

A textbook has to assume some level of prior knowledge--some base on which to build the new subject. This is not at all the same as the COIK principle-- "Clear Only If Known"--which is a trap that most modern texts fall into.　Thus we have (P) in the second paragraph of Chapter 2:

> "People often get the impression that the problem of solving linear systems is much more complex than that of solving nonlinear systems of equations."

Or, consider (H) in the Chapter "Formulas for Definite Integrals."　The third paragraph begins:

> "The point is sometimes raised that when h approaches zero the error term of a high-accuracy formula approaches zero more rapidly than does the error term for a low-accuracy formula.　Unfortunately, the place where this advantage sets in is generally not known, even if the integral is computed at two different spacings, say the second at one-half the spacing of the first."

Or, the treatment in (A) of eigenvalues, beginning with:

> "One of the major problems in the computation laboratory is the determination of the eigenvalues of a matrix.　There are really several related problems, for the matrix may be symmetric and positive definite or it may have some far less obliging form...　In this chapter we shall discuss the relatively simple problem of finding the largest eigen-value of a symmetric matrix by iterative techniques."

It seems to be some kind of an academic in-joke to
assume that the student has been working for years on the
topic at hand, and needs only to be told some of the fine
points.    Unless the students involved are extraordinarily
sophisticated and mature, they are left dangling with
questions like "Just what is the problem we are considering,
and what are some of the ways of solving it?"

First, let us give a quick review of the seven books.

(A)  Most of the material in Acton's book is pure
gold, but it exhibits the COIK principle to an extreme.
For example, the chapter on interpolation never states
what interpolation is, and plunges into Bessel's formula
in a way that no student could follow.   The book
contains much material on practical computing, and Acton
has obviously done a great deal of it.  However, the
material is badly organized; it is presented as a sort
of set of anecdotes, dwelling largely on special patho-
logical cases.   It is a delight for anyone who has done
much numerical work, but students find it baffling.
Perhaps its most serious fault is that it frequently
brings the student to the place where with one or two more
sentences it could explain an important computational
principle, but fails to do so, leaving the uninitiated
student completely in the dark.

(F)  This is the most recent book of the seven.   It
is strangely terse, as though the publisher had specified
that it be held down to 260 pages.   However, the authors
have done much numerical computing, and they include some
tested Fortran programs for specific problem areas.   The
writing is mathematical and scholarly, as though the
material had been prepared for an article in a mathematics
journal.   The discussion on floating arithmetic is very
good, and there is excellent material on instability and
sensitivity of numerical processes.   Forsythe's famous
article on solving a quadratic equation is included.
Again, the experienced worker in numerical methods will
find much of great interest, but a beginning student
would have to have nearly every section translated for him.

(H) This book shows one mathematician at work, with
that passion for elegance and compactness carried to the
point where the material is "Not Clear Even If Known."
There are few examples worked out.   All the right material
is there, but it takes another mathematician to translate
it into usable form for computation.

(M)  This book has the best exposition of the seven.
For every topic, the reader is led gently from what he
already knows to what he is now ready to learn, all done
with a nice style and exceptionally clear English.   The
presence of the computer is felt in the discussion of all
topics.   The chapter on ordinary differential equations
is outstanding, but would be greatly improved if there
were some specific examples worked out for each of the
methods given.   The book has some important topics
buried in exercises, where students will not find them
unless those exercises are specifically assigned.   At
various points in the book, the logic of a method is
explained with "process graphs" which are again COIK,
and which would require long explanations to be meaning-
ful to beginning students.


(P)  A recent book which acknowledges (H) as one of
its models.   The best of the seven for typography and
care in its production.   Many programs--in PL/I--are
presented.   The book stresses the principle that one
should know what one is doing before applying some method
blindly.   Consequently, the material on sketching curves
is excellent.   Much of the material is unique to this
book (e.g., a section on splines).

(S)  A classic; much of it has been in print
continuously for over 50 years.   At the time it was
written there was still not yet a motor-driven calculator
in the world, so some of its material appears quaint.
For every topic, there are carefully worked out examples,
mostly from real-life situations.   It is still one of
the best available books for those who must work with
desk calculators, and it contains excellent examples to
be re-worked today with computers.

(W)  Another classic, somewhat drier than Scarborough,
but with more excellent examples of the application of
each method, all painstakingly worked out, but using
techniques that are not particularly applicable to the
computing world.

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     Suppose we look at one specific topic--finding the
roots of a non-algebraic equation--and see how it is
handled in the seven books.

(A)  Full of good stuff, but on the assumption that you have already devoted a good part of your life to this particular topic.   The examples are contrived and extreme; the usual, normal case is ignored.   However, it is in this part of the book that Acton is at his best in discussing computing strategy and tactics.

(F)  Difficult for a beginner to follow, but the mathematics and computing material is authoritative. Standard methods are presented, followed by a Fortran code for a specific algorithm (ZEROIN) for finding a real zero of a single function which "combines the certainty of bisection with the ultimate speed of the secant method for smooth functions."

(H)  The weakest part of this book.

(M)  Only two methods are presented (Newton-Raphson and successive approximations) and no real examples are worked out.   Rather long winded on theory, and the weakest part of this book on the relation to computers. The chapter's exercises, however, are excellent.

(P)  The only book to lay a proper groundwork for this topic.   The methods of interval-halving, Newton-Raphson, and the secant method are all presented, together with Fortran programs.   Hamming's modified false position method is given here.   A comparison of the various methods is coupled with sound computing advice at the end of the section.   Probably the best treatment of this topic among all the books.

(S)  False position, Newton-Raphson, and an iterative scheme are given, but all with hand calculation in mind.

(W)  Newton-Raphson, false position, and Horner's method are given.   The only book that even mentions Horner's method (and it is good for a book not to). On the other hand, one of the few places where you can find out what Raphson added to Newton's method.


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McCracken and Dorn have a splendid exercise for beginning students:  calculate sine x for x = 30, 390, 750,...,2910 degrees in two ways; by evaluating the Taylor series directly, and by invoking the built-in sine function of whatever language is used on the computer.   Students given this assignment early in a Numerical Methods course (and without too much coaching) will discover many things:

1. A well defined process in Numerical Analysis can yield ridiculous results in computing.

2. The built-in function may not be too trustworthy, either.

3. They are not as adept at coding as they thought they were.

4. A class of 25 students can reach 25 distinct and wildly differing sets of results, even on the same machine using the same language. (A well defined problem, whose solution is run in a controlled atmosphere, ought to produce just one set of results, shouldn't it?)

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Consider a minor topic; namely, Graeffe's root-squaring process, which is not on our list of essential topics, but which is useful and has a long and honorable history. (A) dismisses the process completely, with "a sour note of dissent." (M) has the process buried in an exercise, and poorly explained. (F) and (H) do not mention it at all. (P) comments on the method, but does not demonstrate its use. (S) gives the method a whole chapter, with examples worked out in detail, as does (W). Graeffe's process is not particularly suitable for computer work (unless the process can be continuously monitored, as in a time-sharing system), but it is illustrative of techniques on which many people have devoted considerable effort.

I submit that there is a need for a new Numerical Methods text that is short on derivations and proofs, and long on computing wisdom. For each topic, the best available methods should be presented in a form that can lead to good programs. There can easily be two or more "best" methods, depending on the criteria used, such as ease of understanding, range of applications, modifiability, and so on. The various methods should be compared from the computing point of view; that is, with reference to ease of coding, machine efficiency, limitations due to special cases, convergence rates, and possible pitfalls. The student should be kept in mind at all times--a student who is not a mathematics major and who is not intrigued with elegant notation and overly-abbreviated exposition. He wants, I think, a fairly simple how-to-do-it book that can be used later as a reference manual.

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